A JUMP INVERSION THEOREM FOR THE SEMILATTICES OF SIGMA-DEGREES

ALEXEY STUKACHEV

Abstract. We prove an analogue of the jump inversion theorem for the semilattices of Σ-degrees of structures. As a corollary, we get similar result for the semilattices of degrees of presentability of countable structures.

Keywords: computable structures, effective model theory, generalized computability, admissible sets.

1. Introduction

In these notes, we prove an analogue of the jump inversion theorem for the semilattices of Σ-degrees of structures. The relation of Σ-reducibility, defined on structures of arbitrary cardinality, in the case of countable structures can be viewed as the strongest reducibility in the hierarchy of effective reducibilities [11, 12] (one of the weak reducibilities in this hierarchy is the Muchnik reducibility). In the recent papers, A.A.Soskova and I.N.Soskov [8, 9] proved the jump inversion theorem for the semilattices of degrees of presentability of countable structures with respect to Muchnik reducibility. We will show what the same tools (namely, Marker's extensions, used in computable model theory by S.S.Goncharov and B.Khoussainov [3]) allow to get the same result in the case of Σ-reducibility. As a corollary, we get a jump inversion theorem for all known effective reducibilities on problems of presentability of structures.

For any infinite cardinal α we denote by $\mathcal{K}_\alpha$ the class of structures with cardinality less or equal α, and with finite or computable signatures. For a structure with infinite computable signature we assume what some Gödel numbering of formulas of this signature is fixed.

The next definition is a generalization of one given by Yu.L. Ershov [4], to the case of structures with computable signatures. For simplicity, we give it in the case of a predicate signature.

Definition 1. *Let $\mathfrak{M}$ be a structure of a computable predicate signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots\rangle$ and let $\mathbb{A}$ be an admissible set. Structure $\mathfrak{M}$ is said to be* Σ-definable in $\mathbb{A}$ *if there exists a computable sequence of Σ-formulas*

$$\Phi(x_0, y), \Psi(x_0, x_1, y), \Psi^*(x_0, x_1, y), \Phi_0(x_0, \ldots, x_{n_0-1}, y),$$

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$$\Phi_0^*(x_0,\ldots,x_{n_0-1},y),\ldots,\Phi_k(x_0,\ldots,x_{n_k-1},y),\Phi_k^*(x_0,\ldots,x_{n_k-1},y),\ldots$$

of signature $\sigma_{\mathbb{A}}$ *and a parameter* $a \in A$ *such that, for* $M_0 \rightleftharpoons \Phi^{\mathbb{A}}(x_0,a)$ *and* $\eta \rightleftharpoons \Psi^{\mathbb{A}}(x_0,x_1,a)\cap M_0^2$*, the following holds:* $M_0 \neq \varnothing$*,* η *is a congruence relation on the structure*

$$\mathfrak{M}_0 \rightleftharpoons \langle M_0; P_0^{\mathfrak{M}_0},\ldots,P_k^{\mathfrak{M}_0},\ldots\rangle,$$

where $P_k^{\mathfrak{M}_0} \rightleftharpoons \Phi_k^{\mathbb{A}}(x_0,\ldots,x_{n_k-1})\cap M_0^{n_k}$ *for all* $k\in\omega$*,* $\Psi^{*\mathbb{A}}(x_0,x_1,a)\cap M_0^2 = M_0^2\setminus\Psi^{\mathbb{A}}(x_0,x_1,a)$*,* $\Phi_k^{*\mathbb{A}}(x_0,\ldots,x_{n_k-1},a)\cap M_0^{n_k} = M_0^{n_k}\setminus\Phi_k^{\mathbb{A}}(x_0,\ldots,x_{n_k-1})$ *for all* $k\in\omega$*, and* $\mathfrak{M}$ *is isomorphic to the quotient structure* $\mathfrak{M}_0/\eta$*.*

The relation of Σ-reducibility $\leqslant_\Sigma$ is defined as follows: for structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}\leqslant_\Sigma\mathfrak{B}$ if $\mathfrak{A}$ is Σ-definable in $\mathbb{HF}(\mathfrak{B})$. We assume that the signature of $\mathbb{HF}(\mathfrak{B})$ contains the predicate symbol Sat^2 interpreted as the satisfiability relation for atomic formulas on $\mathfrak{B}$, with respect to the Gödel numbering fixed for the formulas of this signature. In the case of structures with a finite signature this assumption is not essential.

It is easy to check that relation $\leqslant_\Sigma$ is reflexive and transitive. As usual, preordering $\leqslant_\Sigma$ generates on $\mathcal{K}_\alpha$ a relation of Σ-equivalence: $\mathfrak{A}\equiv_\Sigma\mathfrak{B}$ if $\mathfrak{A}\leqslant_\Sigma\mathfrak{B}$ and $\mathfrak{B}\leqslant_\Sigma\mathfrak{A}$. Equivalence classes with respect to $\equiv_\Sigma$ are called *degrees of* Σ*-definability* or Σ*-degrees*. We denote the Σ-degree of a structure $\mathfrak{A}$ by $[\mathfrak{A}]_\Sigma$. The structure

$$\mathcal{S}_\Sigma(\alpha)=\langle\mathcal{K}_\alpha/\equiv_\Sigma,\leqslant_\Sigma\rangle$$

is an upper semilattice with the least element, which is the degree consisting of computable structures. For any structures $\mathfrak{A},\mathfrak{B}\in\mathcal{K}_\alpha$, $[\mathfrak{A}]_\Sigma\vee[\mathfrak{B}]_\Sigma=[(\mathfrak{A},\mathfrak{B})]_\Sigma$, where $(\mathfrak{A},\mathfrak{B})$ is the pair of $\mathfrak{A}$ and $\mathfrak{B}$ in the model-theoretic sense.

The notion of Σ-degree of a structure is invariant from the choice of a semilattice $\mathcal{S}_\Sigma(\alpha)$, because all infinite structures of the same Σ-degree have the same cardinality. There are natural embeddings of semilattices $\mathcal{D}$ of Turing degrees and $\mathcal{D}_e$ of degrees of enumerability of sets of natural numbers into $\mathcal{S}_\Sigma(\omega)$ (and hence into any semilattice $\mathcal{S}_\Sigma(\alpha)$) via the mappings $i:\mathcal{D}\to\mathcal{S}_\Sigma(\omega)$ and $j:\mathcal{D}_e\to\mathcal{S}_\Sigma(\omega)$ defined as follows: for any $\mathbf{a}\in\mathcal{D}$, let $i(\mathbf{a})=[\mathfrak{M}_{\mathbf{a}}]_\Sigma$, where $\mathfrak{M}_{\mathbf{a}}$ is an arbitrary structure with Turing degree $\mathbf{a}$. In the same way, for any e-degree $\mathbf{b}\in\mathcal{D}_e$, let $j(\mathbf{b})=[\mathfrak{M}_{\mathbf{b}}]_\Sigma$, where $\mathfrak{M}_{\mathbf{b}}$ is an arbitrary structure with e-degree $\mathbf{b}$. It is easy to check what these definitions are correct (see [11]). For convenience, we denote $i(\mathbf{a})$ by $\mathbf{a}$.

2. Jump operation for semilattices of Σ-degrees

In all what follows, if not stated otherwise, we consider structures of arbitrary cardinality and with finite or computable signatures. Because we consider structures up to the Σ-equivalence, the following technical proposition allows us to assume they have some additional properties.

Lemma 1. *For any structure* $\mathfrak{A}$ *there is a structure* $\mathfrak{B}\equiv_\Sigma\mathfrak{A}$ *such that*

1) *the signature of* $\mathfrak{B}$ *is finite and does not contain functional symbols;*
2) *the signature of* $\mathfrak{B}$*, together with any predicate symbol* P^k*, contains a predicate symbol* Q^k *such that* $Q^{\mathfrak{B}}=B^k\setminus P^{\mathfrak{B}}$*;*
3) *for any predicate symbol* P^k *from the signature of* $\mathfrak{B}$*, the set* $P^{\mathfrak{B}}$ *is infinite.*

Proof. To prove 1, by substituting the signature functions by the corresponding predicates for their graphs, it is enouth to consider the case when the signature of $\mathfrak{A}$ is infinite and has no functional and constant symbols. Structure $\mathfrak{B}$ can be defined, for example, in the following way. Set as the domain of $\mathfrak{B}$ a disjoint union $A \cup N \cup I$, where A is the domain of $\mathfrak{A}$, N is a countable infinite set, $I = \{i_{\overline{a}} \,|\, \overline{a} \in A^{<\omega}\}$ is a set of cardinality $A^{<\omega}$. The signature $\langle N^1, I^1, S^2, R^4, 0, 1\rangle$ is interpreted on $\mathfrak{B}$ such that $\mathfrak{B} \restriction N \cong \langle \omega, S, 0, 1\rangle$, where S is the succesor relation. Predicate R is defined to be false in all cases except the ones listed below. Namely, for any predicate symbol $R_k^{n_k}$ of the signature of $\mathfrak{A}$ and any tuple $\overline{a} = \langle a_1, \ldots, a_{n_k}\rangle \in A^{n_k}$, $\mathfrak{A} \models R_k(a_1, \ldots, a_{n_k})$ if and only if, in $\mathfrak{B}$ for $s_0, s_1, \ldots, s_k \in N$ such that $s_0 = 0^{\mathfrak{B}}$, $\mathfrak{B} \models S(s_m, s_{m+1})$ for all $m < k$, and for $i = i_{\overline{a}} \in I$, holds

$$\mathfrak{B} \models (R(s_k, a_1, i, 1) \wedge R(a_1, a_2, i, 1) \wedge \ldots \wedge R(a_k, s_k, i, 1)).$$

In the case then $\mathfrak{A} \models \neg R_k(a_1, \ldots, a_{n_k})$ we have the similar definition by substituting in the predicate R element 1 by 0. It is easy to check that $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-equivalent.

Item 2 is evident. To prove 3, assuming 1 and 2 and following [8, 9], consider a Σ-equivalent extension $\mathfrak{B} \supset \mathfrak{A}$ with a disjoint union $A \cup T \cup F$ as the domain, where T and F are infinite, and, for each pair of the "pairwise opposite" in $\mathfrak{A}$ predicate symbols P^k and Q^k and any $\overline{b} \in (A \cup T \cup F)^k$, $\mathfrak{B} \models P(\overline{b})$ if and only if either $\overline{b} \in A^k$ and $\mathfrak{A} \models P(\overline{b})$, or the tuple $\overline{b}$ contains elements from T and no elements from F, and $\mathfrak{B} \models Q(\overline{b})$ if and only if either $\overline{b} \in A^k$ and $\mathfrak{A} \models Q(\overline{b})$, or the tuple $\overline{b}$ contains elements from F. □

Structure $\mathfrak{A}$ is called *$s\Sigma$-definable* in $\mathbb{HF}(\mathfrak{B})$ (denoted by $\mathfrak{A} \subseteq_{s\Sigma} \mathfrak{B}$) if $A \subseteq \mathrm{HF}(B)$ is a Σ-subset in $\mathbb{HF}(\mathfrak{B})$, and all the signature relations and functions of $\mathfrak{A}$ are Δ-definable in $\mathbb{HF}(\mathfrak{B})$. We write $\mathfrak{A} \subset_{s\Sigma} \mathfrak{B}$ to denote the fact that $\mathfrak{A} \subseteq_{s\Sigma} \mathfrak{B}$ and $\mathfrak{B} \not\subseteq_{s\Sigma} \mathfrak{A}$.

The relation of $s\Sigma$-reducibility is analogous to the s-reducibility introduced in [1] for countable structures. Implicitly, the relation of $s\Sigma$-reducibility was introduced in [10] for studying properties of computability over structures (namely, in terms of $s\Sigma$-definability of Skolem extpansions, a criterion for the uniformization property in hereditary finite superstructures over regular structures was introduced in [10]).

For a structure $\mathfrak{A}$, the *jump* of the Σ-degree $[\mathfrak{A}]_\Sigma$ (in $\mathcal{S}_\Sigma(\mathrm{card}(\mathfrak{A}))$) is the Σ-degree of the structure

$$\mathfrak{A}' = (\mathbb{HF}(\mathfrak{A}), \Sigma\text{-Sat}_{\mathbb{HF}(\mathfrak{A})}),$$

where $\Sigma\text{-Sat}_{\mathbb{HF}(\mathfrak{A})}$ is the satisfiability relation of Σ-formulas in $\mathbb{HF}(\mathfrak{A})$. Correctness of this definition follows from the next proposition.

Proposition 1. *For any structures $\mathfrak{A}$ and $\mathfrak{B}$,*

1) *if $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$ then $\mathfrak{A}' \equiv_\Sigma \mathfrak{B}'$;*
2) *$\mathfrak{A} \subset_{s\Sigma} \mathfrak{A}'$;*

Proof. Item 1 follows from the fact that $\mathfrak{A} \leqslant_\Sigma \mathfrak{B}$ implies the existence of an effective interpretation of satisfiability of Σ-formulas in $\mathbb{HF}(\mathfrak{A})$, by means of the Σ-formulas in $\mathbb{HF}(\mathfrak{B})$. Item 2 immediately follows from the fact that in any admissible set there exists a Σ-subset which is not a Δ-subset (see [4]). □

Remark 1. The author doesn't know whether it is possible to put $<_\Sigma$ instead of $\subset_{s\Sigma}$ in the item 2 of the above proposition. Moreover, there exists a countable structure $\mathfrak{A}$ such that $\mathfrak{A} \equiv_w \mathfrak{A}'$, where $\equiv_w$ denotes the Muchnik equivalence (see [11, 5]). Henceforth, the question of existence of fixed points for the Σ-jump operation with respect to the strong reducibilities is the natural one for the future research.

It is easy to verify that the jump operation for Σ-degrees is well defined with respect to the jump operations for Turing and enumeration degrees: if a structure $\mathfrak{A}$ has the Turing (enumeration) degree $\mathbf{a}$ then the structure $\mathfrak{A}'$ has the Turing (enumeration) degree $\mathbf{a}'$.

Remark 2. In a similar way the jump operation was defined in [1] for the semilattice of s-degrees of countable structures. Also, in a similar way the jump operation for admissible sets with respect to different reducibilities was defined in [6, 7].

3. A jump inversion theorem for the semilattices of Σ-degrees

Let $\mathfrak{A}{=}\langle A, R_0^{\mathfrak{A}}, \ldots, R_{n-1}^{\mathfrak{A}}\rangle$ be a structure of a finite relational signature $\sigma{=}\langle R_0^{k_0}, \ldots, R_{n-1}^{k_{n-1}}\rangle$. Following [3], we define *Marker's $\exists$-extension* $\mathfrak{A}^{\exists}$ of $\mathfrak{A}$ as a structure of the signature

$$\sigma^{\exists} = \langle (R_0^{\exists})^{k_0+1}, \ldots, (R_{n-1}^{\exists})^{k_{n-1}+1}, X_0^1, \ldots, X_{n-1}^1\rangle$$

and with the domain $A \cup X_0 \cup \ldots \cup X_{n-1}$, where $A, X_0, \ldots, X_{n-1}$ are disjoint sets and, for all $i < n$,

1) $\mathfrak{A}^{\exists} \models R_i^{\exists}(\overline{a}, x)$ implies $(\overline{a} \in A^{k_i}) \wedge (x \in X_i) \wedge (\mathfrak{A} \models R_i(\overline{a}))$,
2) $(\forall x \in X_i)(\exists! \overline{a} \in A^{k_i})(\mathfrak{A}^{\exists} \models R_i^{\exists}(\overline{a}, x))$,
3) $\mathfrak{A} \models R_i(\overline{a})$ implies $\mathfrak{A}^{\exists} \models (\exists! x \in X_i) R_i^{\exists}(\overline{a}, x)$.

Marker's $\forall$-expansion of a structure $\mathfrak{A}$ is a structure $\mathfrak{A}^{\forall}$ of the signature $\sigma^{\forall}{=}\langle (R_0^{\forall})^{k_0+1}, \ldots, (R_{n-1}^{\forall})^{k_{n-1}+1}, Y_0^1, \ldots, Y_{n-1}^1\rangle$ and with the domain $A \cup Y_0 \cup \ldots \cup Y_{n-1}$, where $A, Y_0, \ldots, Y_{n-1}$ are disjoint sets and, for all $i < n$,

1) $\mathfrak{A}^{\forall} \models R_i^{\forall}(\overline{a}, y)$ implies $(\overline{a} \in A^{k_i}) \wedge (y \in Y_i)$,
2) $(\forall \overline{a} \in A^{k_i})(\exists^{\leqslant 1} y \in Y_i)(\mathfrak{A}^{\forall} \models \neg R_i^{\forall}(\overline{a}, y))$,
3) $(\forall y \in Y_i)(\mathfrak{A}^{\forall} \models R_i^{\forall}(\overline{a}, y))$ implies $\mathfrak{A} \models R_i(\overline{a})$,
4) $(\forall y \in Y_i)(\exists! \overline{a} \in A^{k_i})(\mathfrak{A}^{\forall} \models \neg R_i^{\forall}(\overline{a}, y))$,

(and, hence, $\mathfrak{A} \models R_i(\overline{a})$ if and only if $\mathfrak{A}^{\forall} \models (\forall y \in Y_i) R_i^{\forall}(\overline{a}, y)$).

For a structure $\mathfrak{A}$, onsider the structure

$$\mathfrak{A}^{\circ} = \mathfrak{A}^{\exists\forall} \ (= (\mathfrak{A}^{\exists})^{\forall}).$$

The domain of $\mathfrak{A}^{\circ}$ is the set $A \cup X_0 \cup \ldots \cup X_{n-1} \cup Y_0 \cup \ldots \cup Y_{n-1} \cup Z_0 \cup \ldots \cup Z_{n-1}$, where $X_0, \ldots, X_{n-1}$ are $\exists$-fellows for predicates $R_0^{\mathfrak{A}}, \ldots, R_{n-1}^{\mathfrak{A}}$ of $\mathfrak{A}$, $Y_0, \ldots, Y_{n-1}$ are $\forall$-felows for predicates $(R_0^{\exists})^{\mathfrak{A}^{\exists}}, \ldots, (R_{n-1}^{\exists})^{\mathfrak{A}^{\exists}}$ of $\mathfrak{A}^{\exists}$, and $Z_0, \ldots, Z_{n-1}$ are $\forall$-fellows for predicates $X_0^{\mathfrak{A}^{\exists}}, \ldots, X_{n-1}^{\mathfrak{A}^{\exists}}$ of $\mathfrak{A}^{\exists}$.

The main result of this paper is the following

Theorem 1 (Σ-Jump Inversion)**.** *Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leqslant_{\Sigma} \mathfrak{A}$. There exists a structure $\mathfrak{B}$ for which*

$$\mathfrak{B}' \equiv_{\Sigma} \mathfrak{A}.$$

Proof. We can assume that $\mathfrak{A}$ satisfies the conditions 1, 2, and 3, from Lemma 1. We show that $\mathfrak{A}$ is Σ-equivalent to $\mathfrak{B}'$ for $\mathfrak{B}{=}\mathfrak{A}^{\circ}$.

First, let us show that $\mathfrak{A} \leqslant_\Sigma (\mathfrak{A}^\circ)'$. From the definition of the exentsion $\mathfrak{A}^\circ$ of the structure $\mathfrak{A}$ it follows that the domain of $\mathfrak{A}$ is definable in $\mathfrak{A}^\circ$ by a $\exists\forall$-formula, and hence is Σ-definable in $(\mathfrak{A}^\circ)'$. Second, for any $i < n$ and $\overline{a} \in A^{k_i}$,

$$\mathfrak{A} \models R_i(\overline{a}) \iff \mathfrak{A}^\exists \models (\exists! x \in X_i) R_i^\exists(\overline{a}, x) \iff$$
$$\iff (\mathfrak{A}^\exists)^\forall \models (\exists! x \in X_i)(\forall y \in Y_i) R_i^{\exists\forall}(\overline{a}, x, y).$$

So, the satisfaction of the atomic relations of structure $\mathfrak{A} \subseteq \mathfrak{A}^\circ$ is definable by $\exists\forall$-formula in $\mathfrak{A}^\circ$, and hence is Σ-definable in $(\mathfrak{A}^\circ)'$. As a consequence, $\mathfrak{A} \leqslant_\Sigma (\mathfrak{A}^\circ)'$ (moreover, $\mathfrak{A} \leqslant_{s\Sigma} (\mathfrak{A}^\circ)'$).

Now, let us prove that $(\mathfrak{A}^\circ)' \leqslant_\Sigma \mathfrak{A}$. Immediately from the definition it follows that $\mathfrak{A}^\circ \leqslant_\Sigma \mathfrak{A}$, and henceforth $\mathbb{HF}(\mathfrak{A}^\circ) \leqslant_\Sigma \mathfrak{A}$. Also, it is obvious that predicate

$$\Sigma\text{-Sat}^+_{\mathbb{HF}(\mathfrak{A}^\circ)} = \{\langle \ulcorner \Phi \urcorner, c\rangle \mid \mathbb{HF}(\mathfrak{A}^\circ) \models \Phi(c),\ \Phi(x) \text{ is a } \Sigma\text{-formula}\}$$

is Σ-definable in any Σ-presentation of $\mathbb{HF}(\mathfrak{A}^\circ)$ in $\mathbb{HF}(\mathfrak{A})$ which satisfies an additional assumption from [6]. To prove the Σ-definability of predicate $\Sigma\text{-Sat}_{\mathbb{HF}(\mathfrak{A}^\circ)} \setminus \Sigma\text{-Sat}^+_{\mathbb{HF}(\mathfrak{A}^\circ)}$ it is enough to show that the satisfiability of Π-formulas in $\mathbb{HF}(\mathfrak{A}^\circ)$ is Σ-definable in some Σ-presentation of $\mathbb{HF}(\mathfrak{A}^\circ)$ in $\mathbb{HF}(\mathfrak{A})$.

To describe the subsets definable in $\mathbb{HF}(\mathfrak{A}^\circ)$ by a Π-formulas of signature $\sigma^{\exists\forall} \cup \{\mathrm{U}^1, \in^2\}$ with parameters, for a structure $\mathfrak{M}$ of a signature $\sigma = \langle R_0^{k_0}, \ldots, R_{n-1}^{k_{n-1}}\rangle$ we consider subsets definable in $\mathfrak{M}^\forall$ by a infinite computable conjunctions of the form

$$\bigwedge_{i\in\omega} \forall \overline{u}_i\ \varphi(\overline{v}, \overline{u}_i, \overline{m}_0, \overline{y}_0),$$

where, for all $i \in \omega$, φ_i are quantifier-free formulas of signature $\sigma^\forall$, $\overline{m}_0 \in M^{<\omega}$, $\overline{y}_0 \in (Y_0 \cup \ldots \cup Y_{n-1})^{<\omega}$ are parameters.

We call *positive $\forall$-formula* any formula of the form $\forall\overline{u}\Phi$, where quantifier-free formula Φ is positive, i.e., has no occurences of negation and implication symbols. Relations definable by such formulas (with parameters) are called *positively $\forall$-definable.*

Lemma 2. *The class of $\forall$-definable relations of $\mathfrak{M}^\forall$ consists exactly of the finite unions of positively $\forall$-definable relations in $\mathfrak{M}$ and quantifier-free relations in $\mathfrak{M}^\forall$.*

Proof. We can assume that an $\forall$-definable relation is definable by a $\forall$-formula $\Psi(\overline{v})$ of the form $\forall\overline{u}((P_0 \wedge \ldots \wedge P_k) \rightarrow (Q_0 \vee \ldots \vee Q_l))$, where $P_0, \ldots, P_k, Q_0, \ldots, Q_l$ are atomic formulas (without negations) which have no "wrong" occurences of quantifiers. Note that subformula P_m of the form $Y_i(u_j)$ corresponds to bounding of the quantifier $\forall u_j$ to the set Y_i. We convert the original formula into a formula with bounded quantifiers of the form $(\forall y \in Y_i)$ and eliminate all such quantifiers by the following effective procedure.

It is enough to consider the formula of the form $(\forall y \in Y_i)(S_0 \vee \ldots \vee S_m)$, where $S_0, \ldots, S_m$ are the atomic formulas or their negations, with occurences of variable y. The following procedure of eliminating the bounded quantifier $(\forall y \in Y_i)$ gives a formula equivalent to the original one but with no occurences of y:

1) delete all formulas S_k of the form $R_j^\forall(\overline{u})$ where variable y occurs at the place different from the last one, since any such formula is fasle by the definition of $\forall$-expansion;

2) replace all formulas S_k of the form $\neg R_j^\forall(\overline{u}, y)$ by $R_j(\overline{u})$, according to the definition of the $\forall$-fellow for R_j;

3) remove all formulas S_k of the form $Y_j(y)$, $i \neq j$, and $\neg Y_i(y)$ since they are obviously fasle.

At the end we get a $\forall$-formula without bounded quantifiers and of the form $\forall \overline{u}(S_0 \vee \ldots \vee S_m)$, where $S_0, \ldots, S_m$ are the atomic formulas or their negations, with no subformulas of the form $\neg Y_i$. Further on, for each variable u from the tuple $\overline{u}$ we proceed as follows to prove the lemma:

1) if $Y_{i_0}(u), \ldots, Y_{i_k}(u)$ is the list of all $\forall$-fellows with the occurences of variable u then the corresponding part of the disjunction is non-trivial if and only if $\{i_0, \ldots, i_k\} = \{0, \ldots, n-1\}$. In this case we delete all these subformulas $Y_{i_0}(u), \ldots, Y_{i_k}(u)$;

2) if there are occurences of the variable u in the subformulas $R_i^{\forall}(\overline{v}, u)$ or $\neg R_i^{\forall}(\overline{v}, u)$, we delete these subformulas as false;

3) delete all subformulas of the form $R_i^{\forall}(\overline{v}, y_i^0)$, where $y_i^0 \in Y_i$ but the tuple $\overline{v}$ has occurences of the variables from $\overline{u}$.

□

Lemma 3. *The class of positively $\forall$-definable relations of the structure $\mathfrak{A}^{\exists}$ coincides with the class of relations on $\mathfrak{A}^{\exists}$ which are definable by positive quantifier-free formulas (with parameters).*

Proof. It is suffifient to show that in $\mathfrak{A}^{\exists}$ any formula of the form $\forall u(Q_0 \vee \ldots \vee Q_l)$, where $Q_0, \ldots, Q_l$ are atomic formulas of signature $\sigma^{\exists}$ (with parameters) having occurences of variable u and without occurences of symbols $\neg$ and $=$, is false. We can also assume that $Q_0, \ldots, Q_{n-1}$ are equal, respectively, to the formulas $X_0(u), \ldots, X_{n-1}(u)$, and the formulas $Q_n, \ldots, Q_l$ are different from $X_i(u)$. It is easy to see that, for arbitrary evaluation of the free variables from the formula $\forall u(Q_0 \vee \ldots \vee Q_l)$, the set of values of the variable u for which formula $(Q_n \vee \ldots \vee Q_l)$ is true in $\mathfrak{A}^{\exists}$ is finite (it follows from the definition of the signature predicates of $\mathfrak{A}^{\exists}$). Since the structure $\mathfrak{A}$ is infinite, this completes the proof. □

We are now ready to finish the proof of Theorem 1, using the notions (and notations) from [4] for representations of elements of the hereditary finite superstructures. Any Π-subset $P \subseteq \mathbb{HF}(\mathfrak{A}^{\circ})$ can be represented in the form $P = \cup_{\varkappa \in \mathrm{HF}(\omega)} P_{\varkappa}$, where, for any $\varkappa \in \mathrm{HF}(\omega)$, $P_{\varkappa} = \{\varkappa(\overline{a}) \,|\, \mathfrak{A}^{\circ} \models \Phi_{\varkappa}(\overline{a})\}$, $\Phi_{\varkappa}$ is a computable conjunction of $\forall$-formulas of signature $\sigma^{\exists\forall}$ (with parameters), and $\{\Phi_{\varkappa} \,|\, \varkappa \in \mathrm{HF}(\omega)\}$ is a computable family. Consider a Σ-representation of $\mathbb{HF}(\mathfrak{A}^{\circ})$ in $\mathbb{HF}(\mathfrak{A})$ which satisfies the condition from [6], identical on $\mathfrak{A}$ and in which presentations of the elements of $\mathbb{HF}(\mathfrak{A}^{\circ})$ by terms $\varkappa(\overline{c})$ ($\varkappa \in \mathrm{HF}(\omega)$, $\overline{c} \in |\mathfrak{A}^{\circ}|$) in the sense of [4] is coordinated with some constructivization of $\mathbb{HF}(\omega)$ on ω. By the last condition we mean Δ-definabilty of the relation"$x = \varkappa(\overline{c})$", where x is from the effective presentation of $\mathbb{HF}(\mathfrak{A}^{\circ})$, $\varkappa$ is from the constructive presentation of $\mathbb{HF}(\omega)$, and $\overline{c}$ is a tuple of elements from the presentation of $\mathfrak{A}^{\circ}$. Consider an element $\varkappa(\overline{c}) \in \mathbb{HF}(\mathfrak{A}^{\circ})$, where $\varkappa \in \mathrm{HF}(\omega)$ and $\overline{c} \in |\mathfrak{A}^{\circ}|$ are urelements. Let $\overline{a}$ be all elements of $\overline{c}$ which are from $\mathfrak{A}$ and $\overline{b}$ be all elements of $\overline{c}$ which are from $\mathfrak{A}^{\exists}$. Consider the finite set $T_{\overline{c}}$ formed as the union of the atomic types of $\overline{c}$ in $\mathfrak{A}^{\exists\forall}$, of $\overline{b}$ in $\mathfrak{A}^{\exists}$, and of $\overline{a}$ in $\mathfrak{A}$. Using the effectiveness of the transformations from Lemma 2 and Lemma 3, it is easy to understand that in the Σ-presentation mentioned above the qwestion of satisfyability of a Π-formula on an element of the form $\varkappa(\overline{c})$ is reducible, uniformly and effectively, relative to $T_{\overline{c}}$ (because of the finiteness of the signature there are only finitely many variants for $T_{\overline{c}}$), to the qwestion of satisfyability of a Π-formula on the element $\varkappa$ in $\mathbb{HF}(\omega)$. And the last can be checked effectively in $\mathbb{HF}(\mathfrak{A})$ since $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$.

□

Remark 3. The assumption of finiteness of the signature of $\mathfrak{A}$ which is possible by Lemma 1 is essential for the proof of Theorem 1. In case then the signature of $\mathfrak{A}$ is infinite, reducibility of the original problem to the problem of satisfiability of Π-formulas in $\mathbb{HF}(\omega)$ is impossible since there are infinitely many atomic types of (finite) tuples from $\mathfrak{A}^\circ$.

The jump inversion theorem for the semilattice $\mathcal{S}_\Sigma(\omega)$ implies a similar result for the semilattices of degrees of presentability of countable structures with respect to effective reducibilities. Namely, there is

Corollary 1. *Let $\mathfrak{A}$ be a countable structure such that $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$. Then there is a structure $\mathfrak{B}$ such that, for any $r \in \{e, s, w, ew\}$,*

$$\mathfrak{B}' \equiv_r \mathfrak{A},$$

where e, s, w, ew are the notations for Dyment, Medvedev, Muchnik and non-uniform Dyment reducibilities, respectively.

Proof. Indeed, as it was shown in [11], for any structures $\mathfrak{A}$ and $\mathfrak{B}$, from $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$ it follows that $\mathfrak{A} \equiv_r \mathfrak{B}$ for each $r \in \{e, s, w, ew\}$. □

Remark 4. In [9], besides a spesial case of the above corollary (namely, for $r = w$), in fact, was also proved the following result: the mass problem consisting of the Turing jumps of the presentations of a structure $\mathfrak{A}$ (on natural numbers) is Muchnik equivalent to the problem of presentability of the structure $\mathfrak{A}'$. This result gives one more evidence for the natural choice of $\mathfrak{A}$ as the jump of a structure $\mathfrak{A}$.

Structure $\mathfrak{M}$ is called *locally constructivizable* [4] if the set $\mathrm{Th}_\exists(\mathfrak{M}, \overline{m})$ is computably enumerable for any $\overline{m} \in M^{<\omega}$. Immediately from the proof of Theorem 1 we get some properties of the structure $\mathfrak{B} = \mathfrak{A}^\circ$ which characterize it as "simple" or "low", from the constructive complexity point of view. Namely, there is

Corollary 2. *Let $\mathfrak{A}$ be structure such that $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$. Then there is a locally constructivizable structure $\mathfrak{B}$ such that*

$$[\mathfrak{A}]_\Sigma = [\mathfrak{B}']_\Sigma.$$

Moreover, $\mathfrak{B}' \equiv_\Sigma \mathfrak{B}_1$, where $\mathfrak{B}_1$ is an expansion of $\mathfrak{B}$ by a finite number of first-order definable relations.

Proof. As for the proof of Theorem 1, take $\mathfrak{B} = \mathfrak{A}^\circ$. The equalities $[\mathfrak{B}']_\Sigma = [\mathfrak{B}_1]_\Sigma = [\mathfrak{B}_1]_\Sigma \vee \mathbf{0}'$ follow from the properties of the structure $\mathfrak{A}^\circ$ obtained during the proof of Theorem 1: it is enough to take $\mathfrak{B}_1$ as an expansion of $\mathfrak{B} = \mathfrak{A}^\circ$ by the predicates atomic for structures $\mathfrak{A}$ and $\mathfrak{A}^\exists$. Let us prove the local constructivizability of $\mathfrak{A}^\circ$: suppose that $\overline{c} \in |\mathfrak{A}^\circ|^{<\omega}$ and let $\overline{a}$ are all elements of $\overline{c}$ which are from $\mathfrak{A}$, $\overline{b}$ are all elements of $\overline{c}$ which are from $\mathfrak{A}^\exists$. Consider the finite set $T_{\overline{c}}$ formed as the union of the atomic types of $\overline{c}$ in $\mathfrak{A}^{\exists\forall}$, of $\overline{b}$ in $\mathfrak{A}^\exists$, and of $\overline{a}$ in $\mathfrak{A}$. From the definition of Marker's extensions it immediately follows that the set $\mathrm{Th}_\exists(\mathfrak{A}^\circ, \overline{c})$ is computable relative to $T_{\overline{c}}$.

□

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Alexey Stukachev
Sobolev Institute of Mathematics,
Acad. Koptyug avenue 4,
630090, Novosibirsk, Russia
E-mail address: aistu@math.nsc.ru